

## FUNDAMENTAL GROUPS OF TOPOLOGICAL $R$ -MODULES

BY

ANN BATESON

**ABSTRACT.** The main result of this paper is that if  $R$  is a countable, Noetherian ring, then the underlying abelian group of every  $R$ -module is isomorphic to the fundamental group of some topological  $R$ -module. As a corollary, it is shown that for certain varieties  $V$  (e.g., varieties of finite type) every abelian group in  $V$  is isomorphic to the fundamental group of some arcwise connected topological algebra in  $V$ .

In this paper, we examine the class of abelian groups isomorphic to fundamental groups of topological algebras in a variety  $V$ . Our main result (Theorem 3.1) is that if  $R$  is a countable, Noetherian ring, then the underlying abelian group of every  $R$ -module is isomorphic to the fundamental group of some topological  $R$ -module. Given any variety  $V$ , one can construct a unital ring  $R_V$  such that the abelian groups in  $V$  are precisely the underlying abelian groups of  $R_V$ -modules [see §4]. Hence, we have as a corollary to Theorem 3.1 that if  $R_V$  is a countable, Noetherian ring (e.g.,  $V$  has finite type), every abelian group in  $V$  is isomorphic to the fundamental group of some arcwise connected topological algebra in  $V$ .

In [6], Milnor proves that every countable abelian group is isomorphic to the fundamental group of some topological group. As another consequence of Theorem 3.1, we have that Milnor's result is true for every abelian group regardless of its cardinality.

§1 is devoted to preliminaries, with special emphasis on  $CW$  complexes. In §2 we construct, for every unital ring  $R$  and every countable  $R$ -module  $A$ , a topological  $R$ -module  $T(A)$  with  $\pi_1(T(A))$  isomorphic to  $A$ . We will construct  $T(A)$  as a countable  $CW$  complex. If we try to repeat this construction with an uncountable  $R$ -module, we run into the usual problem that the product topology on  $T(A) \times T(A)$  does not coincide with the  $CW$  topology and  $T(A)$  is not a topological  $R$ -module. Milnor's method in [6] has the same limitation. A common procedure for dealing with this problem with products is to change the topology on the product by working in a "convenient category". Such a procedure would work to extend the results of this section to all  $R$ -modules in the category of compactly generated spaces, but Theorem 3.1 shows that we need not resort to a "convenient category" when  $R$  is a countable, Noetherian ring.

In §3, we prove Theorem 3.1 by using structure theorems for modules over a countable, Noetherian ring. If  $R$  is a countable, Noetherian ring and  $A$  is an  $R$ -module, we first construct a topological  $R$ -module  $T$  with  $\pi_1(T)$  containing the

---

Received by the editors April 14, 1980.

1980 *Mathematics Subject Classification*. Primary 08B99, 22A30; Secondary 57T20, 22A10.

©1982 American Mathematical Society  
0002-9947/81/0000-1005/\$04.25

underlying abelian group of  $A$  as a subgroup.  $T$  is contained in a product  $P$  of countable  $CW$  complexes and we topologize  $P$  with the box topology in order for  $T$  to be a locally simply connected space. We complete the proof of Theorem 3.1 with a covering space argument.

In §4, we examine the implications of Theorem 3.1 for varieties more general than varieties of  $R$ -modules.

This paper evolved from a doctoral thesis submitted to the University of Colorado. I wish to give special thanks to Walter Taylor for his time, inspiration, and generous sharing of ideas. I also thank Sidney Morris for helpful discussions, R. Brown for informing me about "convenient categories", and the referee for suggesting the description of the  $R$ -module  $T(A)$  used in the proof of Theorem 2.1.

**1. Preliminaries and notation.** We first present the basic concepts connected with  $CW$  complexes.

Let  $R$  stand for the set of real numbers and  $R^n$  for real  $n$ -dimensional space. Throughout this paper  $I$  will stand for the unit interval and  $I^n \subseteq R^n$  for the  $n$ -dimensional cube. By a *closed Euclidean  $n$ -cell*, denoted by  $E^n$ , we mean a homeomorphic image of  $I^n$ .  $\dot{E}^n$  will stand for the boundary of  $E^n$  as a subset of  $R^n$ .

**DEFINITION 1.1.** A *cell structure* on a set  $X$  is a pair  $(X, \Phi)$ , where  $\Phi$  is a collection of maps of closed Euclidean cells into  $X$  satisfying:

- (i) if  $\varphi \in \Phi$  and  $\varphi$  has domain  $E^n$ , then  $\varphi$  is injective on  $E^n - \dot{E}^n$ ;
- (ii) the images  $\{\varphi(E^n - \dot{E}^n) \mid \varphi \in \Phi\}$  partition  $X$ ;
- (iii) if  $\varphi \in \Phi$  has domain  $E^n$ , then  $\varphi(\dot{E}^n) \subseteq \bigcup \{\psi(E^k - \dot{E}^k) \mid \psi \in \Phi \text{ has domain } E^k \text{ and } k \leq n-1\}$ .

If  $\varphi \in \Phi$  has domain  $E^n$ , then  $\varphi(E^n) = \sigma^n$  is called an  *$n$ -cell* and we say that  $\varphi$  is a *characteristic map* for  $\sigma^n$ .

**DEFINITION 1.2.** A cell structure is called *closure finite* if each  $n$ -cell  $\sigma^n$  meets only finitely many open cells  $\sigma^p - \dot{\sigma}^p$  with  $p < n$ .

A cell structure is topologized in the following way.

**DEFINITION 1.3.** Let  $(X, \Phi)$  be a cell structure. We define the *weak topology* on  $X$  with respect to  $\Phi$  by:

- (i) each  $n$ -cell  $\sigma$  has the quotient topology induced by  $\varphi_\sigma$  where  $\varphi_\sigma(E^n) = \sigma$ ;
- (ii) a set  $F \subseteq X$  is closed if and only if  $F \cap \sigma$  is closed in  $\sigma$  for each cell  $\sigma$  of  $X$ .

**DEFINITION 1.4.** A space  $X$  is a  *$CW$  complex* if and only if there is a sequence of closed subspaces  $X_0 \subseteq X_1 \subseteq \cdots \subseteq X$  such that  $X = \bigcup X_n$  and

- (i) the set  $X_0$  is discrete;
- (ii) for each  $n$ ,  $X_n$  is obtained from  $X_{n-1}$  by attaching  $n$ -cells;
- (iii) the space  $X$  has the weak topology with respect to the closed sets  $X_n$ .

We conclude with some miscellaneous notation. Let  $X$  be a topological space and  $w, v: I \rightarrow X$  loops based at  $a$ . By  $w \simeq_H v$ , we mean that  $w$  is homotopic to  $v$  relative to  $a$  by a homotopy  $H$ . By  $[w]$  we mean the homotopy equivalence class of loops based at  $a$  containing  $w$  and by  $w * v$  we mean the loop defined by

$$\begin{aligned}(w * v)(t) &= v(2t) \quad \text{if } 0 \leq t \leq \frac{1}{2}, \\ &= w(2t - 1) \quad \text{if } \frac{1}{2} \leq t \leq 1.\end{aligned}$$

All rings will be unital and all  $R$ -modules will be left  $R$ -modules. If  $A$  is an  $R$ -module and  $S$  a submodule of  $A$ , for any  $a, b \in A$ ,  $a \equiv_S b$  means  $a - b \in S$ . If  $T$  is a topological  $R$ -module,  $\pi_1(T)$  will stand for the fundamental group of  $T$  with basepoint 0—i.e.,  $\pi_1(T) = \pi_1(T, 0)$ .

**2. Countable  $R$ -modules.** In this section we prove

**THEOREM 2.1.** *For any countable  $R$ -module  $A$  there exists an arcwise connected topological  $R$ -module  $T(A)$  such that  $A \simeq \pi_1(T(A))$  as  $R$ -modules. Moreover, we can construct  $T(A)$  such that*

- (i) *for every  $a \in A$  there is a loop  $w_a: I \rightarrow T(A)$  such that for every  $r \in R$  and  $t \in I$ ,  $rw_a(t) = w_{ra}(t)$  and the correspondence  $a \rightarrow [w_a]$  is an isomorphism of  $A$  onto  $\pi_1(T(A))$ ;*
- (ii)  *$T(A)$  is a CW complex.*

**PROOF.** Let  $T(A)$  be the  $R$ -module consisting of all  $A$ -valued functions on the unit interval with finite support and vanishing at the endpoints. Let  $a \otimes t$  be the function on the unit interval with value  $a$  at  $t$  and 0 elsewhere. Set  $a \otimes 0 = a \otimes 1 = 0$ . Then, clearly,

**LEMMA 2.1.** *Every element  $f$  of  $T(A)$  can be uniquely written in the form*

$$f = a_1 \otimes t_1 + \cdots + a_n \otimes t_n,$$

*for some nonzero elements  $a_1, \dots, a_n$  of  $A$  and  $t_1, \dots, t_n \in I$  with  $0 < t_1 < \cdots < t_n < 1$ .*

We will call  $a_1 \otimes t_1 + \cdots + a_n \otimes t_n$  as in Lemma 2.1 the canonical representation of  $f$ .

We now define a cell structure on  $T(A)$ . We will have an  $n$ -cell  $\sigma$  for every sequence  $(a_1, \dots, a_n)$  of elements of  $A$  and  $\sigma$  will consist of all elements of the form

$$a_1 \otimes t_1 + \cdots + a_n \otimes t_n$$

where each  $t_i$  varies over  $I$  subject to the condition:  $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq 1$ . We will use the expression “a cell  $\sigma$  is given by  $(a_1, \dots, a_n)$ ” to mean  $\sigma$  consists of the above elements. We define a closed Euclidean cell  $E_\sigma \subseteq [0, 1]^n$  by

$$E_\sigma = \{(t_1, \dots, t_n) \in I: 0 \leq t_1 \leq \cdots \leq t_n \leq 1\},$$

and a characteristic map  $\varphi_\sigma: E_\sigma \rightarrow \sigma$  by

$$\varphi_\sigma(t_1, \dots, t_n) = a_1 \otimes t_1 + \cdots + a_n \otimes t_n.$$

Let  $\Phi^n = \{\varphi_\sigma: \sigma \text{ is a cell given by } (a_1, \dots, a_n)\}$ ,  $\Phi^0 = \{\varphi: 0 \rightarrow 0\}$  and  $\Phi = \bigcup_{n=0}^\infty \Phi^n$ .

**LEMMA 2.2.**  *$(T(A), \Phi)$  is a cell structure.*

**PROOF.** We must show (i)–(iii) of Definition 1.1. Each  $\varphi_\sigma$  is injective on  $E_\sigma - \dot{E}_\sigma$  by Lemma 1.2. Likewise,  $\{\psi_\sigma(E_\sigma - \dot{E}_\sigma): \sigma \text{ varies over the cells of } T(A)\}$  partitions  $T(A)$ . If  $f \in \varphi(\dot{E}_\sigma)$ , then  $f = a_1 \otimes t_1 + \cdots + a_n \otimes t_n$  where some of the  $t_i$ 's are possibly 0 or 1 or equal. In any case, when we reduce  $f$  to its canonical representation,  $f$  belongs to a cell of a lower dimension.

Let  $(T(A), \Phi)$  have the weak topology with respect to  $\Phi$ . To show that  $(T(A), \Phi)$  is a CW complex we need the following two lemmas.

LEMMA 2.3.  $(T(A), \Phi)$  is closure finite.

PROOF. We must show that each  $n$ -cell  $\sigma^n$  meets only finitely many open  $m$ -cells  $\sigma^m - \dot{\sigma}^m$  for  $m < n$ . Suppose

$$\sigma^n \cap (\sigma^m - \dot{\sigma}^m) \neq \emptyset, \quad m < n,$$

and  $\sigma^n$  consists of all elements of the form

$$(1) \quad a_1 \otimes t_1 + \cdots + a_n \otimes t_n,$$

where  $0 \leq t_1 \leq \cdots \leq t_n \leq 1$ . Since there are only finitely many ways (1) can be reduced in length, there are only finitely many possibilities for  $\sigma^m$ .

LEMMA 2.4. The cells of  $(T(A), \Phi)$  are closed subsets of  $T(A)$  with the weak topology.

PROOF. Let  $\sigma$  be a cell given by  $(a_1, \dots, a_n)$  and let  $f \in \dot{\sigma}$ . Let  $x = (s_1, \dots, s_n) \in \dot{E}_\sigma$  with  $\varphi_\sigma(x) = f$ . We define a set  $G_x = \{(t_1, \dots, t_n) \in \dot{E}_\sigma : \text{if } s_i = 0 \text{ or } 1, \text{ then } t_i = s_i \text{ and if } s_i = s_{i+1}, \text{ then } t_i = t_{i+1}\}$ . Note that each  $G_x$  is a closed subset of  $E_\sigma$  and, as  $x$  varies over  $\dot{E}_\sigma$ , there are only finitely many distinct sets  $G_x$ . We claim that  $\varphi_\sigma(G_x) = \tau$ , where  $\tau$  is the cell with  $f \in \tau - \dot{\tau}$ . Assume  $\tau$  is given by  $(b_1, \dots, b_m)$ . Then we may reduce

$$(2) \quad f = a_1 \otimes s_1 + \cdots + a_n \otimes s_n$$

to its canonical representation

$$(3) \quad b_1 \otimes r_1 + \cdots + b_m \otimes r_m$$

where  $(r_1, \dots, r_m) \in E_\tau - \dot{E}_\tau$ . Now we reduce (2) to its canonical representation by deleting all  $a_i \otimes s_i = 0$  and replacing all  $a_i \otimes s_i + a_{i+1} \otimes s_{i+1}$  by  $(a_i + a_{i+1}) \otimes s_i$  if  $s_i = s_{i+1}$ . Hence the sequence of group elements  $(b_1, \dots, b_m)$  depends only on those  $i$  for which either  $s_i = 0$  or 1 or  $s_i = s_{i+1}$ , and not on the particular value of  $s_i$ . Thus  $\varphi_\sigma(G_x) \subseteq \tau$ . To prove the reverse inclusion, we first observe that by the reduction process  $(r_1, \dots, r_m)$  is a subsequence  $(s_{i_1}, \dots, s_{i_m})$  of  $(s_1, \dots, s_n)$  with  $s_{i_k} \neq s_{i_j}$  for  $k \neq j$ . If  $g \in \tau$ ,  $g = b_1 \otimes t'_1 + \cdots + b_m \otimes t'_m$  for some  $(t'_1, \dots, t'_m) \in E_\tau$ . We may embed  $(t'_1, \dots, t'_m)$  into a sequence  $(t_1, \dots, t_n) \in G_x$  with  $t_{i_k} = t'_k$ . Then,  $\varphi_\sigma(t_1, \dots, t_n) = g$  and  $\varphi_\sigma(G_x) = \tau$ .

Let  $\sigma$  be any cell of  $(T(A), \Phi)$ . To show  $\sigma$  is closed we must show that  $\sigma \cap \sigma'$  is closed in  $\sigma'$  for all cells  $\sigma'$ . If  $\sigma \cap \sigma' = \emptyset$  or  $\sigma'$ , we are done. Let  $f \in \sigma \cap \sigma'$ . If  $f \in (\sigma - \dot{\sigma}) \cap (\sigma' - \dot{\sigma}')$ , then  $\sigma = \sigma'$  and  $\sigma \cap \sigma' = \sigma'$ . If  $f \in \dot{\sigma} \cap (\sigma' - \dot{\sigma}')$  and  $\varphi_\sigma(x) = f$ , then  $\varphi_\sigma(G_x) = \sigma' \subseteq \dot{\sigma}$  and again  $\sigma \cap \sigma' = \sigma'$ . Assume  $f \in \dot{\sigma}'$  and let  $\varphi_{\sigma'}(y) = f$ ,  $y \in \dot{E}_{\sigma'}$ . Then  $\varphi_{\sigma'}(G_y) = \tau$ , where  $f \in \tau - \dot{\tau}$ . We look at two cases. If  $f \in \sigma - \dot{\sigma}$ , then  $\sigma = \tau$  and  $\varphi_{\sigma'}(G_y) = \sigma \cap \sigma'$ . If  $f \in \dot{\sigma}$ , let  $\varphi_\sigma(x) = f$ . Then  $\varphi_\sigma(G_x) = \varphi_{\sigma'}(G_y) = \tau$  and  $\varphi_{\sigma'}(G_y) \subseteq \sigma \cap \sigma'$ . Hence, in both cases  $\varphi_{\sigma'}(G_y) \subseteq \sigma \cap \sigma'$  and so  $\varphi_{\sigma'}^{-1}(\sigma \cap \sigma') = \bigcup G_y$ , where the union is taken over all  $y$  such that  $\varphi_{\sigma'}(y) \in \sigma \cap \sigma'$ . Since there are only finitely many sets  $G_y$  and each  $G_y$  is closed in  $E_{\sigma'}$ ,  $\varphi_{\sigma'}^{-1}(\sigma \cap \sigma')$  is a closed subset of  $E_{\sigma'}$ . Since  $\sigma'$  has the quotient topology with respect to  $\varphi_{\sigma'}$ ,  $\sigma \cap \sigma'$  is closed in  $\sigma'$ .

LEMMA 2.5.  $(T(A), \Phi)$  is a CW complex.

PROOF. In Definition 1.4, take each  $X_n$  = the set of cells of  $T(A)$  of dimension  $\leq n$ . It follows from Lemmas 2.3 and 2.4 and from standard arguments that each  $X_n$  is closed. Conditions (i)–(iii) are obvious. For (i), note that  $X_0 = \{0\}$ .

LEMMA 2.6.  $T(A)$  is a topological  $R$ -module.

PROOF. We must show that addition and scalar multiplication are continuous. Since  $A$  is by assumption a countable  $R$ -module,  $T(A)$  is a countable  $CW$  complex, i.e.,  $T(A)$  has only countably many cells. Hence by Theorem II.5.2 in [5], the product space  $T(A) \times T(A)$  is a  $CW$  complex with cells  $\sigma \times \sigma'$ , where  $\sigma$  and  $\sigma'$  are cells of  $T(A)$ . By Proposition II.1.3 [5], it suffices to show that  $+$  restricted to  $\sigma \times \sigma'$  is continuous for all cells  $\sigma, \sigma'$  of  $T(A)$ . It will be convenient to use the symbol  $h$  to stand for the binary operation  $+$ —that is,  $h: T(A) \times T(A) \rightarrow T(A)$  is defined by  $h(a, b) = a + b$ .

Let  $\sigma$  be an  $n$ -cell given by  $(a_1, \dots, a_n)$  and  $\sigma'$  an  $m$ -cell given by  $(a_{n+1}, \dots, a_{n+m})$ . Then  $h(\sigma, \sigma') = \sigma + \sigma'$  is the set of all elements of the form

$$a_1 \otimes t_1 + \dots + a_n \otimes t_n + \dots + a_{n+m} \otimes t_{n+m},$$

where  $0 \leq t_1 \leq \dots \leq t_n \leq 1$  and  $0 \leq t_{n+1} \leq \dots \leq t_{n+m} \leq 1$ .

Let  $E = E_\sigma \times E_{\sigma'}$ . For each permutation  $\pi$  of  $\{1, \dots, m+n\}$  such that  $\pi(j) = i$  and  $\pi(l) = k$ , where  $1 \leq i \leq k \leq n$  or  $n+1 \leq i \leq k \leq n+m$ , implies  $j \leq l$ , we define a set

$$H_\pi = \{(x_1, \dots, x_{m+n}) \in E: 0 \leq x_{\pi(1)} \leq \dots \leq x_{\pi(m+n)} \leq 1\}.$$

Note that  $H_\pi$  is a closed subset of  $E$ , there are only finitely many sets  $H_\pi$  and  $E = \bigcup H_\pi$ . Let  $\sigma_\pi$  be the cell given by  $(a_{\pi(1)}, \dots, a_{\pi(m+n)})$ . Then  $H_\pi$  is homeomorphic to  $E_{\sigma_\pi}$  by a homeomorphism  $\lambda: E_{\sigma_\pi} \rightarrow H_\pi$ . If  $\varphi: E \rightarrow \sigma \times \sigma'$  is the characteristic map for  $\sigma \times \sigma'$ ,  $\varphi_{\sigma_\pi} = (h\varphi)\lambda$ . Since  $\varphi_{\sigma_\pi}$  is continuous and  $\lambda$  is a homeomorphism,  $h\varphi \upharpoonright H_\pi$  is continuous for each  $\pi$ .

Let  $F$  be a closed subset of  $T(A)$ . We will show that  $h^{-1}(F)$  is closed. Now  $\varphi^{-1}h^{-1}(F) = \bigcup (\varphi^{-1}h^{-1}(F) \cap H_\pi)$  and each  $\varphi^{-1}h^{-1}(F) \cap H_\pi$  is closed in  $H_\pi$ , and hence in  $E$ , since  $h\varphi \upharpoonright H_\pi$  is continuous. Since there are only finitely many sets  $H_\pi$ ,  $\varphi^{-1}h^{-1}(F)$  is closed in  $E$ . Since  $\sigma \times \sigma'$  has the quotient topology with respect to  $\varphi$ ,  $h^{-1}(F)$  is closed in  $\sigma \times \sigma'$ .

To show scalar multiplication is continuous, first note that for each  $r \in R$  and each cell  $\sigma$  given by  $(a_1, \dots, a_n)$ , if  $(a_{i_1}, \dots, a_{i_k})$  is the largest subsequence of  $(a_1, \dots, a_n)$  such that for each  $j = 1, \dots, k$ ,  $ra_{i_j} \neq 0$ , then  $r\sigma = \{rf: f \in \sigma\}$  is the  $k$ -cell given by  $(ra_{i_1}, \dots, ra_{i_k})$ . We have the commutative diagram

$$\begin{array}{ccc} E_\sigma & \xrightarrow{p} & E_{r\sigma} \\ \varphi_\sigma \downarrow & & \downarrow \varphi_{r\sigma} \\ \sigma & \xrightarrow{\bar{r}} & r\sigma \end{array}$$

where  $p$  is the projection  $p(t_1, \dots, t_n) = (t_{i_1}, \dots, t_{i_k}) \in E_{r\sigma}$ , and  $\bar{r}(f) = rf, f \in \sigma$ . If  $H$  is a closed subset of  $r\sigma$ , then  $\varphi_\sigma^{-1}\bar{r}^{-1}(H) = p^{-1}\varphi_{r\sigma}^{-1}(H)$  is closed in  $E_\sigma$  since  $p$  and

$\varphi_{r\sigma}$  are continuous. Since  $\sigma$  has the quotient topology with respect to  $\varphi_\sigma$ ,  $\bar{r}^{-1}(H)$  is closed in  $\sigma$ . Hence scalar multiplication restricted to each cell  $\sigma$  is continuous, whence by Proposition II.1.3 [5] it is continuous on  $T(A)$ .

From Lemmas 2.5 and 2.6 we know that  $T(A)$  is a  $CW$  complex and a topological  $R$ -module. Hence we need only prove (i) of Theorem 2.1 and that  $A \simeq \pi_1(T(A))$  as  $R$ -modules. Associate with each  $a \in A$  the loop  $w_a$  in  $T(A)$  defined by  $w_a(t) = a \otimes t$  for each  $t$  in the unit interval. Since  $w_a = \varphi_\sigma$  where  $\sigma$  is given by  $(a)$ , it is continuous. Clearly, for every  $r \in R$  and  $t \in I$ ,  $rw_a(t) = w_{ra}(t)$  in  $T(A)$  and we will show that the correspondence  $a \rightarrow [w_a]$  is a group isomorphism of  $A$  onto  $\pi_1(T(A))$ . Since  $T(A)$  is a  $CW$  complex, the 1-cells of  $(T(A), \Phi)$  give the generators of  $\pi_1(T(A))$  and the 2-cells give the relations. Since each 1-cell  $\sigma'$  is given by  $(a)$  for some  $a \in A$ ,  $\pi_1(T(A))$  consists of the homotopy classes  $[w_a]$ ,  $a \in A$ . Each 2-cell  $\sigma^2$  is given by some  $(a, b)$ ,  $a, b \in A$ , and consists of elements of the form

$$a \otimes t_1 \oplus b \otimes t_2$$

where  $0 \leq t_1 \leq t_2 \leq 1$ . Note that  $E_{\sigma^2}$  is a triangle with edges  $e_1 = I \times \{1\}$ ,  $e_2 = \{0\} \times I$  and  $e_3 = \{(t, t) : t \in I\}$ .  $\varphi_{\sigma^2} \upharpoonright e_1$  traces out  $w_a$ ,  $\varphi_{\sigma^2} \upharpoonright e_2$  traces out  $w_b$  and  $\varphi_{\sigma^2} \upharpoonright e_3$  traces out  $w_{a+b}$ . But since each  $a \otimes t + b \otimes t = (a+b) \otimes t$  for all  $t \in I$ ,  $\varphi_{\sigma^2} \upharpoonright e_3$  traces out  $w_{a+b}(t)$  and the relations of  $\pi_1(T(A), 0)$  are

$$[w_a] + [w_b] = [w_{a+b}],$$

whence  $a \rightarrow [w_a]$  is an isomorphism of  $A$  onto  $\pi_1(T(A))$ .

We can define a scalar multiplication on  $\pi_1(T(A), 0)$  by  $r[w_a] = [rw_a] = [w_{ra}]$ . This scalar multiplication makes  $\pi_1(T(A), 0)$  into an  $R$ -module and  $A$  and  $\pi_1(T(A))$  are isomorphic as  $R$ -modules.

### 3. Modules over a countable, Noetherian ring. We use Theorem 2.1 to prove

**THEOREM 3.1.** *If  $R$  is a countable, Noetherian ring, then the underlying abelian group  $A_G$  of every  $R$ -module  $A$  is isomorphic to  $\pi_1(T(A))$  for some arcwise connected topological  $R$ -module  $T(A)$ .*

The link between Theorems 2.1 and 3.1 is the following lemma concerning the structure of modules over a countable, Noetherian ring.

**LEMMA 3.1.** *If  $R$  is a countable, Noetherian ring, then every  $R$ -module can be embedded into a direct sum of countable  $R$ -modules.*

**PROOF.** Assume  $R$  is a countable, Noetherian ring. Every  $R$ -module can be embedded into an injective  $R$ -module [3, p. 80] and since  $R$  is Noetherian, every injective  $R$ -module is isomorphic to a direct sum of uniform injective  $R$ -modules [3, p. 103]. (An  $R$ -module is uniform if and only if each of its submodules is indecomposable.) Hence, we need only show that every uniform injective  $R$ -module is countable. Now a uniform injective  $R$ -module is isomorphic to the injective hull of  $R/L$  for some meet irreducible left ideal  $L$  of  $R$  [3, p. 100]. Since  $R$  is countable, so is  $R/L$ . Let  $A_0 = R/L$  and let  $E$  be the injective hull of  $A_0$ . We will show that  $E$  is countable.

Since  $E$  is injective, given any left ideal  $L'$  of  $R$  and any homomorphism  $\alpha: L' \rightarrow A_0$ ,  $\alpha$  has an extension to a homomorphism  $\alpha': R \rightarrow E$ . For each such  $\alpha$ , fix an extension  $\alpha'$ . Let

$$A_1 = A_0 + \left( \sum_{L' \text{ an ideal of } R, \alpha': L' \rightarrow A_0} \alpha'(R) \right),$$

where  $+$  and  $\Sigma$  indicate the sum, not necessarily direct, of the submodule  $A_0$  and the submodules  $\alpha'(R)$ . Then,  $A_1$  is a submodule of  $E$  such that every homomorphism  $\alpha: L' \rightarrow A_0$  has an extension to a homomorphism  $\alpha': R \rightarrow A_1$ . Since  $R$  is a countable Noetherian ring and  $A_0$  is a countable  $R$ -module,

- (i) each  $\alpha'(R)$  is countable,
- (ii) the number of left ideals  $L'$  of  $R$  is countable, and
- (iii) for each left ideal  $L$ , the number of homomorphisms  $\alpha: L' \rightarrow A_0$  is countable.

Hence,  $A_1$  is a countable submodule of  $E$ .

Repeating the construction of  $A_1$  from  $A_0$ , each time using  $A_i$  in place of  $A_0$  to construct  $A_{i+1}$ , we generate a nest of countable submodules of  $E$

$$A_0 \subseteq A_1 \subseteq \cdots \subseteq A_i \subseteq A_{i+1} \cdots$$

such that for each left ideal  $L'$  and each homomorphism  $\alpha: L' \rightarrow A_i$ ,  $\alpha$  has an extension to a homomorphism  $\alpha': R \rightarrow A_{i+1}$ . Then,  $A = \bigcup A_i$  is a countable submodule of  $E$ . We claim  $A$  is injective. Let  $\alpha: L' \rightarrow A$  be a homomorphism and  $L'$  a left ideal of  $R$ . Since  $R$  is Noetherian,  $L'$  is finitely generated, whence  $\alpha(L') \subseteq A_i$  for some  $i$ . But then  $\alpha$  has an extension to a homomorphism  $\alpha': R \rightarrow A_{i+1} \subseteq A$ . That  $A$  is injective follows from the Injective Test Lemma [3, p. 73]. By the minimality of the injective hull  $E \supseteq A \supseteq A_0$ ,  $E = A$  and so  $E$  is countable.

For the rest of this section  $R$  will stand for a countable, Noetherian ring.

**PROOF OF THEOREM 3.1.** Let  $A$  be an arbitrary  $R$ -module. Then, by Lemma 3.1, there exist countable  $R$ -modules  $A_\lambda$ ,  $\lambda \in \Lambda$ , such that  $A$  can be embedded into  $\bigoplus A_\lambda$  as a submodule. For each  $\lambda \in \Lambda$ , let  $T(A_\lambda)$  be the topological  $R$ -module of Theorem 2.1. In particular, the underlying abelian group of  $A_\lambda$  is isomorphic to  $\pi_1(T(A_\lambda))$ .

We look at the product  $\prod T(A_\lambda)$  with the box topology—i.e., sets of the form  $\prod G_\lambda$ ,  $G_\lambda$  open in  $T(A_\lambda)$ , form a basis for the topology. We will use the notation  $\prod_B T(A_\lambda)$  to indicate the box topology. We will write elements of a product  $\prod X_\lambda$  as  $\langle x_\lambda \rangle$ ,  $x_\lambda \in X_\lambda$  for each  $\lambda$ . We will use the symbol  $\rho_\lambda$  to stand for the projection of  $\prod X_\lambda$  onto  $X_\lambda$ . If  $f: X \rightarrow \prod X_\lambda$ ,  $f_\lambda = \rho_\lambda \circ f$ . If  $f_\lambda: X \rightarrow X_\lambda$ ,  $\langle f_\lambda \rangle: X \rightarrow \prod X_\lambda$  is the function defined by  $\langle f_\lambda \rangle_\lambda = f_\lambda$ . Given an element  $\langle x_\lambda \rangle \in \prod X_\lambda$ , by the expression “ $x_\lambda$  has property  $P$  almost everywhere” (abbreviated a.e.), we mean  $x_\lambda$  has this property for all but finitely many  $\lambda$ . Similarly, if  $f: X \rightarrow \prod X_\lambda$ , for “ $f$  has a property  $P$  a.e.” we mean  $f_\lambda$  has this property for all but finitely many  $\lambda$ . We will use the following remarks about the box topology.

**REMARK 3.1.** Each projection  $\rho_\lambda: \prod_B X_\lambda \rightarrow X_\lambda$  is continuous, and hence if  $f: X \rightarrow \prod_B X_\lambda$  is continuous, so is each  $f_\lambda: X \rightarrow X_\lambda$ .

**REMARK 3.2.** If  $f: X \rightarrow \prod_B X_\lambda$  is such that  $f_\lambda$  is identically constant a.e. and each  $f_\lambda$  is continuous, then  $f$  is continuous.

PROOF. If  $\prod_B G_\lambda$  is a basic open set,  $f^{-1}(\prod_B G_\lambda) = \cap f_\lambda^{-1}(G_\lambda)$ , which is open since each  $f_\lambda^{-1}(G_\lambda)$  is open and  $f_\lambda^{-1}(G_\lambda) = X$  or  $\emptyset$  a.e.

LEMMA 3.2.  $\prod_B T(A_\lambda)$  is a topological  $R$ -module.

PROOF. Immediate from the fact that each  $T(A_\lambda)$  is a topological  $R$ -module.

We restrict our attention to the direct sum  $\oplus_B T(A_\lambda)$ , topologized as a subspace of  $\prod_B T(A_\lambda)$ . Since  $\prod_B T(A_\lambda)$  is a topological  $R$ -module and  $\oplus_B T(A_\lambda)$  is a submodule of  $\prod_B T(A_\lambda)$ ,  $\oplus_B T(A_\lambda)$  is also a topological  $R$ -module.

Since  $A$  can be embedded into  $\oplus A_\lambda \subseteq \prod A_\lambda$ , we will express each  $a \in A$  as a sequence  $\langle a_\lambda \rangle$ ,  $a_\lambda \in A_\lambda$  with  $a_\lambda = 0$  a.e.. Let  $w_{a_\lambda}: I \rightarrow T(A_\lambda)$  be the loop representing  $a_\lambda$  in Theorem I (ii). If  $a_\lambda = 0$ ,  $w_{a_\lambda} \equiv 0$ . Let  $w_a = \langle w_{a_\lambda} \rangle$ . Then, since  $w_{a_\lambda} \equiv 0$  a.e.,  $w_a: I \rightarrow \oplus_B T(A_\lambda)$  and, by Remark 3.2,  $w_a$  is continuous. Let

$$\pi_0 = \{[w_a]: a \in A\} \subseteq \pi_1\left(\oplus_B T(A_\lambda)\right).$$

Let  $\varphi: A \rightarrow \pi_0$  be defined by  $\varphi(a) = [w_a]$ . Then,

LEMMA 3.3.  $\pi_0$  is a subgroup of  $\pi_1(\oplus_B T(A_\lambda))$  and  $\varphi: A_G \rightarrow \pi_0$  is an isomorphism of  $A_G$  onto  $\pi_0$ , where  $A_G$  is the underlying abelian group of the  $R$ -module  $A$ .

PROOF. Since  $\varphi(A) = \pi_0$ , it suffices to show that  $\varphi$  is a monomorphism. Let  $a = \langle a_\lambda \rangle$  and  $b = \langle b_\lambda \rangle$ . Then  $\varphi(a) * \varphi(b) = [\langle w_{a_\lambda} \rangle] * [\langle w_{b_\lambda} \rangle] = [\langle w_{a_\lambda} * w_{b_\lambda} \rangle]$ . By Theorem 2.1, for each  $\lambda$ ,  $w_{a_\lambda} * w_{b_\lambda}$  is homotopic to  $w_{a_\lambda + b_\lambda}$  in  $T(A_\lambda)$  by a homotopy  $H_\lambda$ . Since  $w_{a_\lambda} \equiv w_{b_\lambda} \equiv 0$  a.e.,  $H_\lambda$  may be taken to be identically 0 a.e. Hence by Remark 3.2,  $\langle H_\lambda \rangle$  is continuous and  $[\langle w_{a_\lambda} * w_{b_\lambda} \rangle] = [\langle w_{a_\lambda + b_\lambda} \rangle]$ . That is,  $\varphi(a) * \varphi(b) = \varphi(a + b)$ , or  $\varphi$  is a homomorphism. If  $\varphi(a) = \varphi(b)$ , then  $\langle w_{a_\lambda} \rangle$  is homotopic to  $\langle w_{b_\lambda} \rangle$  by a homotopy  $H$ . By Remark 3.1, each  $H_\lambda: I \times I \rightarrow T(A_\lambda)$  is continuous, whence, for each  $\lambda$ ,  $w_{a_\lambda}$  is homotopic to  $w_{b_\lambda}$ ,  $a_\lambda = b_\lambda$  and so  $a = b$ .

To construct a covering space  $T(A)$  of  $\oplus_B T(A_\lambda)$  with fundamental group isomorphic to  $\pi_0$  we need that  $\oplus_B T(A_\lambda)$  is an arcwise connected, locally arcwise connected, and locally simply connected space. Since each  $T(A_\lambda)$  is arcwise connected, it follows from Remark 3.2 that  $\oplus_B T(A_\lambda)$  is arcwise connected. We prove that  $\oplus_B T(A_\lambda)$  is locally arcwise connected and locally simply connected by proving the stronger result in Lemma 3.5. For the proof of Lemma 3.5, we need

LEMMA 3.4. For every vertex  $a$  of a CW complex  $X$  and for every open set  $V$  with  $a \in V$ , there exists an open set  $U$ ,  $a \in U \subseteq V$ , and a strong deformation retraction  $p$  of  $U$  onto  $a$  such that for any open set  $G$ ,  $a \in G \subseteq U$ , there exists an open set  $H$ ,  $a \in H \subseteq G$ , and  $p(H \times I) \subseteq H$ .

PROOF. We review the proof of Theorem II.6.1. [5, p. 63], in which an open set  $U \subseteq V$  and a strong deformation retraction  $p$  of  $U$  onto  $a$  are constructed. The existence of an open set  $H$  with  $a \in H \subseteq G \subseteq U$  and  $p(H \times I) \subseteq H$  follows immediately from an examination of this proof.

For each  $n$ -cell  $\sigma$  of  $X$ , fix a characteristic map  $\varphi_\sigma: D^n \rightarrow \sigma$  where  $D^n$  is the Euclidean  $n$ -disc.  $X^n$  will denote the  $n$ -dimensional skeleton of  $X$  and  $C^n$  will denote  $C \cap X^n$ , where  $C \subseteq X$ .



$U$  and  $p$  are constructed by recursively constructing each  $U^n$  and a strong deformation retraction  $p^n$  of  $U^n$  onto  $a$ . For  $n = 0$ , let  $U^0 = a$  and define  $p^0$  by  $p^0(a, t) = a$ ,  $0 \leq t \leq 1$ . Suppose for each  $k \leq n$  we have constructed  $U^k$  and a strong deformation retraction  $p^k$  of  $U^k$  onto  $a$  such that

1.  $U^k$  is open in  $X^k$ ,  $\bar{U}^k \subseteq V^k$ , where  $\bar{U}^k$  is the closure of  $U^k$ , and  $U^k \cap X^{k-1} = U^{k-1}$ , and
2.  $p^k(x, t) = x$  for  $0 \leq t \leq 2^{-k}$  and  $p^k \upharpoonright U^{k-1} \times I = p^{k-1}$ .

We construct  $U^n$  and  $p^n$  with these same properties.

For each  $n$ -cell  $\sigma$  whose interior meets  $V$ , we look at a characteristic map  $\varphi_\sigma: D^n \rightarrow \sigma$ . Note that  $\varphi_\sigma^{-1}(U^{n-1}) \subseteq \dot{D}^n$ , where  $\dot{D}^n$  is the boundary of  $D^n$ . Let  $\delta$  equal the distance between  $\varphi_\sigma^{-1}(U^{n-1})$  and  $D^n - \varphi_\sigma^{-1}(V)$  and let

$$W_\sigma^n = \{(1-s)z: z \in \varphi_\sigma^{-1}(U^{n-1}) \text{ and } 0 \leq s \leq \delta/4\}.$$

We have  $\overline{W_\sigma^n} \subseteq V^n$ . Let  $U_\sigma^n = \varphi_\sigma(W_\sigma^n)$  and  $U^n = U^{n-1} \cup (\cup_\sigma U_\sigma^n)$ . Then  $U^n \subseteq \bar{U}^n \subseteq V^n$  and one may easily verify that  $U^n$  is open in  $X^n$ .

Let  $D^n - \{0\}$  stand for the  $n$ -disc with the center removed. Define  $r^n: (D^n - \{0\}) \times [0, 2^{-n+1}] \rightarrow D^n \times \{0\}$  by

$$\begin{aligned} r^n(x, t) &= x \quad \text{for } 0 \leq t \leq 2^{-n}, \\ r^n(x, t) &= (2 - 2^n t)x + (2^n t - 1) \left( \frac{x}{\|x\|} \right) \quad \text{for } 2^{-n} \leq t \leq 2^{-n+1}. \end{aligned}$$

That is,  $r^n$  is a radial deformation retraction of  $D^n - \{0\}$  onto  $\dot{D}^n$ . If  $\sigma$  is a  $n$ -cell,  $W_\sigma^n \subseteq D^n - \{0\}$  and  $r^n(W_\sigma^n \times [0, 2^{-n+1}]) \subseteq W_\sigma^n$ . Let  $r_\sigma^n = r^n \upharpoonright W_\sigma^n \times [0, 2^{-n+1}]$ . Note that  $r_\sigma^n$  is a radial deformation retraction of  $W_\sigma^n$  onto  $\varphi_\sigma^{-1}(U^{n-1})$ .

We define a strong deformation retraction  $p^n$  of  $U^n$  onto  $a$  by

$$\begin{aligned} p^n(x, t) &= \varphi_\sigma r_\sigma^n(\varphi_\sigma^{-1}(x), t) \quad \text{for } x \in \sigma, \quad 0 \leq t \leq 2^{-n+1}, \\ p^n(x, t) &= p^{n-1}(p^n(x, 2^{-n+1}), t) \quad \text{for } 2^{-n+1} \leq t \leq 1. \end{aligned}$$

Then,  $p^n$  has the properties in (ii) above and our induction is completed. We take  $p = \cup p^n$  and  $U = \cup U^n$ .

Given an open set  $G$  with  $a \in G \subseteq U$ , we may replace  $V$  by  $G$  in the above construction to obtain an open set  $H$ ,  $a \in H \subseteq G$ , and a strong deformation retraction  $p_H$  of  $H$  onto  $a$ . However, examining the above construction of  $p$ , we see that  $p$  is constructed from the characteristic maps  $\varphi_\sigma$  and the radial deformation retractions  $r^n$  and these maps do not depend upon the open set  $V$ , whence  $p_H = p \upharpoonright H \times I$  and so  $p(H \times I) \subseteq H$ .

**LEMMA 3.5.** *For every  $a \in \bigoplus_B T(A_\lambda)$  and every open set  $V$  containing  $a$  there is an open set  $U$ ,  $a \in U \subseteq V$ , and a strong deformation retraction  $p$  of  $U$  onto  $a$ . Hence,  $\bigoplus_B T(A_\lambda)$  is locally arcwise connected and locally simply connected.*

**PROOF.** Since  $\bigoplus_B T(A_\lambda)$  is a topological group, we may assume without loss of generality that  $a = 0$  and  $V$  is a neighborhood of 0. Hence  $V$  contains an open set of the form  $\prod_B V_\lambda \cap \bigoplus_B T(A_\lambda)$  where for each  $\lambda$ ,  $0 \in V_\lambda$  and  $V_\lambda$  is open in  $T(A_\lambda)$ . Each  $T(A_\lambda)$  is a  $CW$  complex. Let  $U_\lambda \subseteq V_\lambda$ ,  $0 \in U_\lambda$ , be the open set of Lemma 3.4

and  $p_\lambda$  the strong deformation retraction of  $U_\lambda$  onto 0. Let  $U = \prod_B U_\lambda \cap \bigoplus_B T(A_\lambda) \subseteq \prod_B V_\lambda \cap \bigoplus_B T(A_\lambda)$ . Define a map  $p: U \times I \rightarrow U$  by

$$p(\langle x_\lambda \rangle, t) = \langle p_\lambda(x_\lambda, t) \rangle$$

for every  $\langle x_\lambda \rangle \in U$  and  $t \in I$ . Since  $x_\lambda = 0$  a.e.,  $p_\lambda(x_\lambda, t) = 0$  a.e. and  $p(\langle x_\lambda \rangle, t) \in U$ . We claim that  $p$  is a strong deformation retraction of  $U$  onto 0.

We need only show that  $p$  is continuous. Take any  $y \in U$  and an open set  $G = \prod G_\lambda \cap \bigoplus_B T(A_\lambda)$  with  $y \in G$ . Without loss of generality, we may assume  $G_\lambda \subseteq U_\lambda \subseteq V_\lambda$ . Suppose  $p(x, t) = y$ , where  $x = \langle x_\lambda \rangle$ ,  $x_\lambda = 0$  a.e. For each  $\lambda$  such that  $x_\lambda = 0$ , appealing to Lemma 3.4, we select an open set  $H_\lambda \subseteq G_\lambda$  with  $0 \in H_\lambda$  such that  $p_\lambda(H_\lambda \times I) \subseteq H_\lambda \subseteq G_\lambda$ . Assume only  $x_{\lambda_1}, \dots, x_{\lambda_m}$  are unequal to 0. Since each  $p_{\lambda_i}$ ,  $i = 1, \dots, m$ , is continuous, there is an open set  $H_{\lambda_i}$  of  $T(A_{\lambda_i})$  with  $x_{\lambda_i} \in H_{\lambda_i}$  and an open set  $I_{\lambda_i}$  of  $I$  with  $t \in I_{\lambda_i}$  such that  $p(H_{\lambda_i}, I_{\lambda_i}) \subseteq G_{\lambda_i}$ . Let  $H = \prod H_\lambda \cap \bigoplus_B T(A_\lambda)$  and  $I' = \bigcap_{i=1}^m I_{\lambda_i}$ . Then  $(x, t) \in H \times I'$  and  $p(H \times I') \subseteq G$ , whence  $p$  is continuous.

Since  $\bigoplus_B T(A_\lambda)$  is arcwise connected and locally simply connected, we can construct the covering space  $T(A)$  of  $\bigoplus_B T(A_\lambda)$  with fundamental group isomorphic to  $\pi_0$  and hence to  $A_G$ . Since  $\bigoplus_B T(A_\lambda)$  is a topological group, so is  $T(A)$  [4, p. 217]. We complete the proof of Theorem 3.1 by showing that  $T(A)$  is a topological  $R$ -module.

We define scalar multiplication in  $T(A)$  in the obvious way. Recall that the elements of  $T(A)$  are equivalence classes  $\{w\}_{\pi_0}$  of paths  $w$  with  $w(0) = 0$ . Two paths  $w$  and  $v$  are equivalent if there exists a loop  $u$ ,  $[u] \in \pi_0$ , with  $w \approx_H u * v$ .

For each equivalence class  $\{w\}_{\pi_0}$  and for each  $r \in R$ , we define  $r \cdot \{w\}_{\pi_0} = \{rw\}_{\pi_0}$ , where  $rw$  is the path given by  $(rw)(t) = r(w(t))$  for all  $t \in I$ . We verify that multiplication is well defined. Suppose  $w \simeq_H u * v$ ,  $\{u\}_H \in \pi_0$ . Then

$$rw \simeq_H r(u * v) = ru * rv.$$

Since  $\{u\} \in \pi_0$ ,  $u$  is homotopic to a loop  $\langle w_{a_\lambda} \rangle$  where  $\langle a_\lambda \rangle \in A_G$  and each  $w_{a_\lambda}$  is the loop representing  $a_\lambda$  as in Theorem 2.1 (i). Hence

$$\begin{aligned} ru * rv &\simeq_H r \langle w_{a_\lambda} \rangle * rv \\ &= \langle rw_{a_\lambda} \rangle * rv \\ &= \langle w_{ra_\lambda} \rangle * rv. \end{aligned}$$

Since  $\langle w_{ra_\lambda} \rangle \in \pi_0$ ,  $rv \in \{rw\}_{\pi_0}$ .

The proof that scalar multiplication is continuous is routine and left to the reader.

We can define scalar multiplication on  $\pi_0 = \{[w_a]: a \in A\} \subseteq \pi_1(\bigoplus_B T(A_\lambda), 0)$  by  $r[w_a] = [w_{ra}]$ . Let  $\tilde{w}_a$  be  $w_a$  lifted to  $T(A)$ . Since  $\pi_1(T(A)) = \{[\tilde{w}_a]: a \in A\}$ , we can likewise define scalar multiplication on  $\pi_1(T(A))$  in such a way that

**COROLLARY 3.1.**  $\pi_1(T(A))$  and  $A$  are isomorphic as  $R$ -modules.

Taking  $R = \mathbb{Z}$  in Theorem 3.1, we have

**COROLLARY 3.2.** Every abelian group is isomorphic to the fundamental group of some topological group.

**4. Applications to Universal Algebra.** Theorem 3.1 has applications to algebraic structures other than  $R$ -modules and, in fact, was inspired by Walter Taylor's paper *Varieties obeying homotopy laws* [7]. In this section we define some terms from Universal Algebra which we will use to describe these applications.

An *algebra*  $\mathfrak{A}$  of type  $(n_t)_{t \in T}$  consists of a nonempty set  $A$  together with a family  $(f_t)_{t \in T}$  of operations such that  $f_t: A^{n_t} \rightarrow A$  for each  $t \in T$ . By a *variety*  $V$ , we mean a class of algebras for which there exists a set  $\Sigma$  of equations such that  $\mathfrak{A} \in V$  if and only if  $e$  holds true in  $\mathfrak{A}$  for every  $e \in \Sigma$ . For example, one speaks of the variety of groups, the variety of  $R$ -modules, and the variety of lattices. By a *topological algebra in a variety*  $V$ , we mean a structure  $(A, \mathfrak{T}, f_t)_{t \in T}$ , where  $(A, \mathfrak{T})$  is a Hausdorff space,  $(A, f_t)_{t \in T} \in V$ , and each  $f_t: A^{n_t} \rightarrow A$  is continuous with the product topology on  $A^{n_t}$ .

A *groupoid* is a small category in which every morphism is an isomorphism. A *groupoid in a variety*  $V$  is a structure  $(G, D, \cdot, ^{-1}, f_t)_{t \in T}$  such that  $(G, f_t)_{t \in T} \in V$ ,  $(G, D, \cdot, ^{-1})$  is a groupoid, where  $D \subseteq G \times G$  is the domain of the binary operation  $\cdot$ , and  $D \rightarrow G$  and  $G \rightarrow G$  are  $V$ -homomorphisms (i.e., they commute with the operations  $f_t, t \in T$ ). A *group in a variety*  $V$  is a one object groupoid in  $V$ .

Let  $V$  be a variety and let  $\Pi_V = \{G: G \text{ is a group isomorphic to the fundamental group of some arc-component of some topological algebra in } V\}$ . Let  $\mathcal{G}_V = \{G: G \text{ is the automorphism group of some object of some groupoid in } V\}$ .

Taylor proves in [7] that  $\Pi_V \subseteq \mathcal{G}_V$  and the group laws obeyed by all groups in  $\Pi_V$  coincide with those laws obeyed by all groups in  $\mathcal{G}_V$ . Whether or not  $\Pi_V = \mathcal{G}_V$  remains an open question. However, as a corollary to Theorem 3.1, we prove (Theorem 4.1) that under certain conditions  $\mathcal{Q}_V \subseteq \Pi_V$  where  $\mathcal{Q}_V \subseteq \mathcal{G}_V$  consists of all abelian groups in  $V$ .

In [7], Taylor mentions the previously known fact that for each variety  $V$  there is a unital ring  $R_V$  such that the abelian groups in  $V$  are precisely the underlying abelian groups of  $R_V$ -modules. For any variety  $V$ ,  $R_V$  is constructed by taking noncommuting free generators  $\alpha_f^1, \dots, \alpha_f^n$  for each  $n$ -ary operation  $f$  and setting

$$R_V = Z[\alpha_f^i: \text{all } f, \text{ all } i]/\Sigma,$$

where  $Z$  is the ring of integers and  $\Sigma$  is an ideal constructed as follows. For each  $n$ -ary operation  $f$ , write

$$f(x_1, \dots, x_n) = \alpha_f^1 x_1 + \dots + \alpha_f^n x_n.$$

Let  $w_1$  and  $w_2$  be the coefficients of an arbitrary variable  $x_i$  in terms  $\tau_1$  and  $\tau_2$ , where  $\tau_1 = \tau_2$  is an equation of  $V$ . Then  $\Sigma$  is generated by all such  $w_1 - w_2$ . (For more details, see [7, p. 521].) For example, let  $V$  be the variety defined by the equations

$$(*) \quad f(x, x) = x, \quad f(f(x, y), x) = y.$$

Let  $G$  be an abelian group in  $V$ . Since  $f: G^2 \rightarrow G$  is a group homomorphism, we have endomorphisms  $\alpha_i: G \rightarrow G, i = 1, 2$ , such that

$$f(x, y) = \alpha_1(x) + \alpha_2(y).$$

The equations  $(*)$  imply

$$(**) \quad \alpha_1 + \alpha_2 = 1, \quad \alpha_1^2 + \alpha_2 = 0, \quad \alpha_1 \alpha_2 = 1.$$

Let  $\Sigma$  be the ideal generated by  $\alpha_1 + \alpha_2 - 1$ ,  $\alpha_1^2 + \alpha_2$ , and  $\alpha_1\alpha_2 - 1$ . Then  $R_V = Z[\alpha_1, \alpha_2]/\Sigma$ . Clearly, every abelian group in  $V$  is an  $R_V$ -module and given any  $R_V$ -module  $G$ , we can use  $\alpha_1$  and  $\alpha_2$  to define a homomorphism  $f: G^2 \rightarrow G$  satisfying (\*). Incidentally, since  $\alpha_1 = (1 + \sqrt{3}i)/2$  and  $\alpha_2 = (1 - \sqrt{3}i)/2$  satisfy (\*\*),  $R_V = Z[(1 + \sqrt{3}i)/2]$ .

If  $T$  is a topological  $R_V$ -module for any variety  $V$ , we can use  $\alpha_1^{f_t}, \dots, \alpha_{n_t}^{f_t}$  to define the  $n_t$ -ary operation  $f_t$  on the underlying topological space of  $T$  and  $(T, f_t)_{t \in T}$  is a topological algebra in  $V$ . Hence, it follows from Theorem 3.1 that

**THEOREM 4.1.** *If  $V$  is a variety such that  $R_V$  is a countable, Noetherian ring, then every abelian group in  $V$  is isomorphic to the fundamental group of some arcwise connected topological algebra in  $V$ .*

If  $V$  is a variety of finite type, then  $R_V$  is countable and, by Hilbert's basis theorem, Noetherian, whence

**THEOREM 4.2.** *If  $V$  is a variety of finite type, then every abelian group in  $V$  is isomorphic to the fundamental group of some arcwise connected topological algebra in  $V$ .*

In [7] Taylor proves that if every group in  $V$  obeys a group law  $\lambda$  and  $(A, \mathfrak{T}, f_t)_{t \in T}$  is a topological algebra in  $V$  with one element subalgebra  $\{a\}$ , then  $\pi_1(A, a)$  obeys  $\lambda$ . From Theorem 4.1, we can prove the partial converse.

**COROLLARY 4.1.** *Let  $V$  be a variety such that  $R_V$  is a countable, Noetherian ring. Suppose that for every topological algebra  $(A, \mathfrak{T}, f_t)_{t \in T}$  in  $V$  with one element subalgebra  $\{a\}$ ,  $\pi_1(A, a)$  obeys  $\lambda$ . Then every abelian group in  $V$  obeys  $\lambda$ .*

**PROOF.** Let  $G$  be an abelian group in  $V$ . Then, by Theorem 4.1,  $G \simeq \pi_1(A, a)$  where  $(A, \mathfrak{T}, f_t)_{t \in T}$  is a topological algebra in  $V$  and  $a \in A$ .  $(A, \mathfrak{T}, f_t)_{t \in T} \in V$  is constructed from an  $R_V$ -module whose underlying set is  $A$  and  $a$  is the zero of  $A$  as an  $R$ -module, whence  $f_t(a, \dots, a) = a$  for every  $t \in T$  and  $\{a\}$  is a one element subalgebra of  $A$ . By assumption  $\pi_1(A, a)$  obeys  $\lambda$ , whence  $G$  obeys  $\lambda$ .

## REFERENCES

1. H. Dowker, *Topology of metric complexes*, Amer. J. Math. **74** (1952), 555–577.
2. C. Faith and E. Walker, *Direct-sum representation of injective modules*, J. Algebra **5** (1967), 203–221.
3. T. Head, *Modules: a primer of structure theorems*, Brooks/Cole, Monterey, California, 1947.
4. P. J. Hilton and S. Wylie, *Homology theory*, Cambridge Univ. Press, New York, 1960.
5. A. T. Lundell and S. Weingram, *The topology of CW complexes*, Van Nostrand Reinhold, New York, 1969.
6. J. Milnor, *The geometric realization of a semisimplicial complex*, Ann. of Math. **65** (1957), 357–362.
7. W. Taylor, *Varieties obeying homotopy laws*, Canad. J. Math. **3** (1967), 498–527.

*Current address:* 819 Spruce Street, Boulder, Colorado 80302